A. Arbitrary Waves

It is not difficult to show what kinds of measurements are necessary and how they should be interpreted in order to deduce directly the stress, density, and energy states obtaining in an arbitrary wave. This method requires no assumptions about the form of the constitutive relation, steadiness of the wave, or extent to which the states are equilibrium states.

The equations expressing conservation of mass, momentum and energy are, in Lagrangian coordinates:[38]

$$(\partial V/\partial t) - V_0(\partial u/\partial h) = 0 \tag{7}$$

$$(\partial u/\partial t) + V_0(\partial P/\partial h) = 0 \tag{8}$$

$$(\partial E/\partial t) + (P/\rho_0)(\partial u/\partial h) = 0 \tag{9}$$

In these equations h is the initial (undisturbed) coordinate of a particle, ρ is density (ρ_0 is initial density), P is the stress component in the direction of propagation, u is mass velocity, and E is specific internal energy.

These equations are generally valid. The only restriction is that heat flow has been assumed negligible in Eq. (9). Thus, within that restriction, the equations apply equally to compressive or release waves and are independent of any assumption about the constitutive relation.

We define two phase velocities associated with the wave as follows:

$$c_u = (\partial h/\partial t)_u$$

and

$$c_p = (\partial h/\partial t)_p$$

since h and t are the independent variables, P = P(h,t) and u = u(h,t), and,

$$dP = (\partial P/\partial t)dt + (\partial P/\partial h)dh,$$

also,

$$du = (\partial u/\partial t)dt + (\partial u/\partial h)dh.$$

Thus,

$$c_p = -\frac{\partial P/\partial t}{\partial P/\partial h} \tag{10}$$

$$c_u = -\frac{\partial u/\partial t}{\partial u/\partial h} \tag{11}$$

Combining Eqs. (7) and (11):

$$(\rho_o/\rho^2)(\partial\rho/\partial t) - (1/c_u)(\partial u/\partial t) = 0$$

or

$$(\rho_o/\rho^2)d\rho - (1/c_u)du = 0, \quad \text{along } h = \text{const.}$$

Combining Eqs. (8) and (10):

$$(\partial u/\partial t) - (1/\rho_o c_p)(\partial P/\partial t) = 0$$

or

$$dP = \rho_o c_p \, du, \quad \text{along } h = \text{const.}$$

Similarly:

$$dE = (P/\rho_o c_u)du, \quad \text{along } h = \text{const.}$$

Summarizing, we have

$$dV = -du/\rho_o c_u \tag{12}$$